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1999 J. Phys. A: Math. Gen. 32 5355

(http://iopscience.iop.org/0305-4470/32/28/313)

J. Phys. A: Math. Gen. **32** (1999) 5355–5366. Printed in the UK PII: S0305-4470(99)98399-3

The inverse variational problem for autoparallels

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Received 12 October 1998, in final form 11 February 1999

Abstract. We study the problem of the existence of a local quantum scalar field theory in a general affine metric space that in the semiclassical approximation would lead to the autoparallel motion of wavepackets, thus providing a deviation of the spinless particle trajectory from the geodesics in the presence of torsion. The problem is shown to be equivalent to the inverse problem of the calculus of variations for the autoparallel motion with the additional conditions that the action (if it exists) has to be invariant under time reparametrizations and general coordinate transformations, while depending analytically on the torsion tensor. The problem is proved to have no solution for a generic torsion in four-dimensional spacetime. A solution exists only if the contracted torsion tensor is a gradient of a scalar field, while the traceless part is zero. The corresponding field theory describes coupling of matter to the dilaton field.

1. Introduction and motivations

In Riemann–Cartan spaces, a connection $\Gamma_{\mu\nu}{}^{\sigma}$ compatible with the metric $g_{\mu\nu}$ (meaning that $D_\mu g_{\nu\sigma} = 0$, with D_μ being the covariant derivative) may have non-vanishing antisymmetric components $S_{\mu\nu}{}^{\sigma} = \frac{1}{2}(\Gamma_{\mu\nu}{}^{\sigma} - \Gamma_{\nu\mu}{}^{\sigma})$ which are the torsion tensor components in a coordinate basis. A general affine connection compatible with the metric can always be represented in the form [1] $\Gamma_{\mu\nu}{}^{\sigma} = \overline{\Gamma}_{\mu\nu}{}^{\sigma} + g^{\sigma\alpha}(S_{\mu\nu\alpha} - S_{\nu\alpha\mu} + S_{\alpha\mu\nu})$, where $\overline{\Gamma}_{\mu\nu}{}^{\sigma}$ are the Christoffel symbols associated with the metric $g_{\mu\nu}$. As was first pointed out by Cartan, the existence of connections that are compatible with the metric and do not coincide with the natural Riemannian connection $\overline{\Gamma}_{\mu\nu}{}^{\sigma}$ may lead to more general theories of gravity than Einstein's general relativity (see, e.g., [2] for a review and references therein). Consequently, the actual motion of a spinless point particle may, in principle, deviate from the usual geodesic motion due to an interaction with torsion.

The torsion force cannot be arbitrary and its possible form should be obtained from some physical principles. It is natural to assume the actual motion of a particle to enjoy general coordinate covariance. A trajectory of the motion is determined by its tangent vector (or velocity). To specify the corresponding equations of motion, one has to define a variation of the velocity along the trajectory. In a space with a general affine connection there exist two independent variation operators that involve a displacement and produce tensors out of tensors (i.e. variations covariant under general coordinate transformations): the Lie derivative and the covariant derivative [1], p 335. A physically acceptable variation should contain the displacement $d_u u^\mu = \dot{u}^\mu$ of the velocity along itself (acceleration). The Lie

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0305-4470/99/285355+12$30.00

derivative does not provide us with such a displacement. Therefore, the only possibility is

$$D_u u^\mu = u^\nu D_\nu u^\mu = \dot{u}^\mu + \Gamma_{\nu\sigma}{}^\mu u^\nu u^\sigma = F^\mu(S, g, u) \tag{1.1}$$

where F^μ is a vector force. Next we require that the motion becomes geodesic when the torsion vanishes, that is, the vector u^μ is transported parallel along itself with respect to a natural Riemannian connection $\overline{\Gamma}_{\mu\nu}{}^\sigma$

$$\overline{D}_u u^\mu = \dot{u}^\mu + \overline{\Gamma}_{\nu\sigma}{}^\mu u^\nu u^\sigma = 0. \tag{1.2}$$

This leads to the condition $F^\mu(S = 0, g, u) = 0$. The simplest possibility proposed first by Ponomarev [3] is to set $F^\mu = 0$. The corresponding curve is called the autoparallel. Its characteristic geometrical property is similar to that of geodesics. The tangent vector is transported parallel along itself with respect to a full affine connection. However, it does not share another property of geodesics such as being the shortest curve between two points of the manifold.

As follows from the comparison of equations (1.2) and (1.1) with $F^\mu = 0$, the deviation of the autoparallel from the geodesic is caused by the torsion force $2S_{\mu\nu\sigma}u^\nu u^\sigma$. The choice between the geodesic and the autoparallel motion can either be decided experimentally or on theoretical grounds following from the compatibility of the postulate $F^\mu = 0$ in (1.1) with other fundamental principles of physics. In [4] it is argued that the energy–momentum conservation law of a spinless point particle leads to geodesics rather than to autoparallels. The conclusion is based on the earlier work by Papapetrou [5] that prescribes a specific relation between the canonical momentum and the velocity of the particle. In general, the energy–momentum tensor is defined as the variational derivative of the particle Lagrangian with respect to the metric tensor. Its conservation law specifies the particle equations of motion that are the usual Euler–Lagrange equations. Hence, if equation (1.1) admits the Euler–Lagrange form, then the energy–momentum conservation law may lead to the autoparallels as is shown in appendix A with an explicit example.

Based on a physical analogy between spaces with torsion and crystals with topological defects [6], the attention has been brought again to the autoparallel motion in [7], section 10, where it was also quantized by the path integral method. The approach gives a consistent (non-relativistic) quantum theory only for a special ('gradient') torsion [7], section 11. For a generic torsion it leads to difficulties with the probabilistic interpretation of the corresponding quantum mechanics and does not comply with the correspondence principle [8].

The problem of coupling between matter and the spacetime geometry is undoubtedly of great importance. So far only the principle of minimal gauge coupling has been explored [2, 9], except, maybe, for the conformal coupling [10]. Based on the minimal gauge coupling principle and the conservation theorems it has been argued [11] that in two possible generalizations of Einstein's general relativity, known as the tetrad theory and the Poincaré gauge theory, scalar matter has the same classical equations of motion as in Einstein's theory. It is noteworthy that the renormalization arguments in quantum field theory with an external torsion background have led the authors of [12] to the conclusion that *non-minimal* coupling is necessary to make the theory consistent. The aim of the present work is to approach the problem from a different and more general point of view, where no assumption about a form of the theory of gravity with torsion nor about the conservation laws is made. The idea is as follows. All models of the fundamental interactions are described by quantum field theory. Thus, if the autoparallels indeed describe the motion of a spinless point particle in a general Riemann–Cartan space, then they must follow from a local quantum scalar field theory in the semiclassical (eikonal) approximation. A conventional way to construct a quantum field theory that satisfies the correspondence principle is first to quantize the relativistic particle motion, thus obtaining relativistic quantum mechanics, and then to apply the so-called second quantization procedure [13].

Consider, for example, the geodesic motion (1.2) which follows from a least action principle for the action

$$S_g = \int L_g\, \mathrm{d}t = -m \int \sqrt{g_{\mu\nu} v^\mu v^\nu}\, \mathrm{d}t = -m \int \mathrm{d}s \tag{1.3}$$

where $v^\mu = \mathrm{d}q^\mu/\mathrm{d}t$. In (1.1) it has been set $\dot{u}^\mu = \mathrm{d}u^\mu/\mathrm{d}s$ and $u^\mu = \mathrm{d}q^\mu/\mathrm{d}s$. To quantize the system, one goes over to the canonical Hamiltonian formalism by means of the Legendre transformation for v^μ. Defining the canonical momentum $p_\mu = \partial L_g/\partial v^\mu$ we find that the canonical Hamiltonian $H = p_\mu v^\mu - L_g = 0$ vanishes identically. This happens due to the local time reparametrization symmetry of the action (1.3). It is not hard to be convinced that the Hessian $H_{\mu\nu} = \partial^2 L_g/(\partial v^\mu \partial v^\nu)$ is degenerate (in particular, $H_{\mu\nu} v^\nu = 0$) and, therefore, the system has a constraint. It has the well known form $\Pi = p^2 - m^2 = 0$. According to Dirac [14], after promoting p_μ and q^μ to self-adjoint operators satisfying the Heisenberg algebra, the constraint $\hat{\Pi}$ has to annihilate physical states

$$\hat{\Pi}\psi = (\hat{p}^2 - m^2)\psi = 0 \tag{1.4}$$

where $-\hat{p}^2$ is the Laplace–Beltrami operator ($\hbar = 1$). In doing so, we obtain a relativistic quantum mechanics that leads to the geodesic motion of the wavepackets in the eikonal approximation. We remark that there is an operator ordering ambiguity upon quantization $p^2 \to \hat{p}^2$, and, in general, the operator $\hat{\Pi}$ may have corrections (of order $\hbar^2$) proportional to the scalar curvature [10]. However, they are not relevant for the leading order of the eikonal approximation. The canonical Hamiltonian vanishes identically, hence, the Schrödiger evolution $\mathrm{i}\partial_t \psi = \hat{H}\psi \equiv 0$ is trivial. Therefore the constraint (1.4) entirely specifies the evolution of relativistic quantum particle states. This latter property allows one to construct a corresponding quantum field theory. If all solutions of (1.4) are labelled by a set of parameters k, then a Heisenberg quantum field operator that carries quanta (particles) with quantum numbers k and wavefunctions $\psi_k(q)$ reads $\hat{\phi} = \sum_k \psi_k(q)\hat{a}_k + \text{h.c.}$ where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are destruction and creation operators of these quanta. The corresponding action of such a field theory in n dimensions is [10]

$$S = \int \mathrm{d}^n q\, \sqrt{g}\, \phi \hat{\Pi} \phi = \int \mathrm{d}^n q\, \sqrt{g} \big(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2\big). \tag{1.5}$$

Thus the constraint occurring through the time reparametrization (gauge) symmetry specifies the sought-for quantum field theory obeying the correspondence principle.

The same strategy can be applied to build a relativistic quantum theory for the autoparallel motion. That is, we need a Lagrangian for equation (1.1). It has to fulfil some additional physical conditions: (a) to be time reparametrization invariant; (b) to be invariant under general coordinate transformations (i.e. to be a scalar); and (c) to turn into (1.3) as the torsion approaches zero (analyticity in torsion). We remark that the autoparallel equation (1.1) with $F_\mu \equiv 0$ exhibits the time reparametrization symmetry therefore it is natural to expect the Lagrangian to fulfil condition (a). Yet, as has been pointed out, the constraint occurring through this gauge symmetry entirely determines the evolution of a *relativistic* quantum particle interacting with the spacetime geometry. The second condition is the standard one: physics cannot depend on the choice of a coordinate system. The third one is natural since we expect a small deviation from the geodesic motion in the limit of small torsion. Thus, we have reduced our problem to the well known and, in fact, long-standing problem of mathematical physics. Given a set of equations of motion, find out whether they admit the Euler–Lagrange form. This is the inverse problem of the calculus of variations. Necessary and sufficient conditions for the solution to exist were first formulated by Helmholtz [15].

2. The Helmholtz conditions for the autoparallel motion

Let the equations of motion be a system of differential equations of second order

$$G_\mu(\dot v, v, q) = H_{\mu\nu}(v, q)\dot v^\nu + B_\mu(v, q) = 0. \tag{2.1}$$

The question arises: does there exist a Lagrangian whose Lagrange derivative $[L]_\mu$ coincides with the equation of motion? That is,

$$G_\mu = [L]_\mu \equiv \frac{\partial^2 L}{\partial v^\mu \partial v^\nu}\dot v^\nu + \frac{\partial^2 L}{\partial v^\mu \partial q^\nu} v^\nu - \frac{\partial L}{\partial q^\mu}. \tag{2.2}$$

Helmholtz found necessary and sufficient conditions on the functions G_μ of the *independent* variables q, v, $\dot v$ in order for the Lagrangian to exist [15]:

$$\frac{\partial G_\mu}{\partial \dot v^\nu} = \frac{\partial G_\nu}{\partial \dot v^\mu} \tag{2.3}$$

$$\frac{\partial G_\mu}{\partial v^\nu} + \frac{\partial G_\nu}{\partial v^\mu} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial G_\mu}{\partial \dot v^\nu} + \frac{\partial G_\nu}{\partial \dot v^\mu}\right\} \tag{2.4}$$

$$\frac{\partial G_\mu}{\partial q^\nu} - \frac{\partial G_\nu}{\partial q^\mu} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial G_\mu}{\partial v^\nu} - \frac{\partial G_\nu}{\partial v^\mu}\right\}. \tag{2.5}$$

With respect to an arbitrary time parameter t the autoparallel equation (1.1) is

$$G_\mu = [L_g]_\mu + 2S_{\mu\nu\lambda}\frac{v^\nu v^\lambda}{\sqrt{v^2}} = 0. \tag{2.6}$$

The geodesic term $[L_g]_\mu$ obviously fulfils the Helmholtz conditions. The second term in (2.6) is the torsion force that causes a deviation of the trajectory from the geodesics $[L_g]_\mu = 0$. Due to the linearity in G_μ, the Helmholtz conditions yield restrictions on the torsion force only. From the second Helmholtz condition (2.4), the restriction $S_{\mu(\nu\lambda)} = 0$ on torsion can be deduced. This implies vanishing of the torsion force in (2.6). Thus, equation (2.2) does *not* have any solution for a *non-vanishing* torsion force.

The only possibility to find a Lagrangian formalism for the autoparallel is to look for an *equivalent* set of equations which may have the Euler–Lagrange form. This can be done by introducing a multiplier $\Omega_\mu{}^\nu(v, q)$ with $\det \Omega_\mu{}^\nu \neq 0$ which acts as an integrating factor in equation (2.2). We are then looking for a solution to the equation

$$[L]_\mu = \Omega_\mu{}^\nu G_\nu. \tag{2.7}$$

The integrability conditions (2.3)–(2.5) become less restrictive for G_μ itself since some of them can be fulfilled by an appropriate choice of the multipliers. This procedure was first proposed in [16]. Although there has been much progress in this approach (see [17]) and some useful techniques have been invented to simplify the Helmholtz conditions, the problem still remains unsolved in general. Recently, the inverse variational problem for equation (1.1) with $F_\mu = 0$ has been solved in two dimensions [18]. However, in these works the proper time s in the equation of motion (1.1) has been considered as the Lagrangian time t. The actions obtained are not time reparametrization invariant. Consequently, it would be difficult to give them a physical interpretation in the framework of a relativistic theory. However, they might be useful to study a non-relativistic autoparallel motion on two-dimensional surfaces.

3. The gradient case

Here we give an example where the Helmholtz integrability conditions are fulfilled for the generalized problem (2.7). Consider a special case where the trace of the torsion tensor is a gradient and the traceless part vanishes,

$$S_{\mu\nu}{}^{\lambda} = \tfrac{1}{2}\big(\delta^{\lambda}_{\mu}\,\partial_{\nu}\sigma - \delta^{\lambda}_{\nu}\,\partial_{\mu}\sigma\big). \tag{3.1}$$

The corresponding autoparallel equation (2.6) follows from the least action principle $\delta S_{(\sigma)} = 0$ where [19] (see also [20])

$$S_{(\sigma)} = \int L_{(\sigma)}\,\mathrm{d}t = -m\int \mathrm{e}^{\sigma(q)}\sqrt{g_{\mu\nu}v^{\mu}v^{\nu}}\,\mathrm{d}t = -m\int \mathrm{e}^{\sigma(q)}\,\mathrm{d}s. \tag{3.2}$$

Whereas the action (1.3) for geodesics is just an integral over proper time, in (3.2) a scalar factor $\mathrm{e}^{\sigma(q)}$ occurs. The same Lagrangian was obtained in Brans–Dicke theory [21], where the masses of particles depend on position $m \to m(q) = m\mathrm{e}^{\sigma(q)}$. The scalar field σ can also be interpreted as the dilaton field [22] emerging in the low-energy limit of the string theory together with the metric $g_{\mu\nu}$.

The Lagrange derivative of $L_{(\sigma)}$ reads

$$[L_{(\sigma)}]_{\mu} = \mathrm{e}^{\sigma}\left([L_g]_{\mu} + (g_{\mu\lambda}\partial_{\nu}\sigma - g_{\nu\lambda}\partial_{\mu}\sigma)\frac{v^{\nu}v^{\lambda}}{\sqrt{v^2}}\right) = 0. \tag{3.3}$$

It has the form (2.7) with the multiplier $\Omega_{\mu}{}^{\nu} = \mathrm{e}^{\sigma}\delta^{\nu}_{\mu}$. Equation (3.3) exhibits the time reparametrization symmetry. The trajectory can be defined in a gauge-invariant way by specifying the proper time. Since the theory has an extra scalar function σ available, the gauge-invariant time is not unique: $\mathrm{d}s = f(\sigma)\sqrt{g_{\mu\nu}v^{\mu}v^{\nu}}\,\mathrm{d}t$ with $f(\sigma)$ being a general positive function of σ. If we set $f = 1$, equation (3.3) turns into the autoparallel equation

$$g_{\mu\nu}\dot{u}^{\nu} + (\overline{\Gamma}_{\lambda\nu\mu} + g_{\mu\lambda}\partial_{\nu}\sigma - g_{\nu\lambda}\partial_{\mu}\sigma)u^{\lambda}u^{\nu} = 0. \tag{3.4}$$

It should be stressed that the trajectory depends on the definition of the (proper) gauge-invariant time. For instance, with the choice $f = \mathrm{e}^{\sigma}$ equation (3.3) turns into a *geodesic* equation. Indeed, under the conformal transformation,

$$g_{\mu\nu} \longrightarrow g^{(\sigma)}_{\mu\nu} = \mathrm{e}^{2\sigma}g_{\mu\nu} \tag{3.5}$$

the action (3.2) goes over to the action (1.3) for geodesics associated with the new metric $g^{(\sigma)}_{\mu\nu}$ and the new proper time $\mathrm{d}s^{(\sigma)} = \mathrm{e}^{\sigma}\,\mathrm{d}s$. Thus, a violation of Einstein's equivalence principle due to the 'dilaton' force in (3.4) can be observed, provided there is a possibility to distinguish experimentally between the measurements of distances and time intervals relative to the metrics $g_{\mu\nu}$ and $g^{(\sigma)}_{\mu\nu}$. We return to this issue later in the conclusions. The metric rescaling (3.5) can be used to remove the force caused by the 'gradient' part of the torsion tensor from the equation of motion:

$$G_{\mu}(g_{\alpha\beta}, S_{\alpha\beta\gamma}) = \mathrm{e}^{\sigma}G_{\mu}\big(\mathrm{e}^{-2\sigma}g_{\alpha\beta}, S_{\alpha\beta\gamma} + S^{(\sigma)}_{\alpha\beta\gamma}\big) \tag{3.6}$$

where $S^{(\sigma)}_{\alpha\beta\gamma}$ is given by (3.1) and in both sides of equation (3.6) the proper time is defined with $f = 1$.

Now we make use of this symmetry to built up a quantum field theory which in a semiclassical approximation would lead to the autoparallel motion of the wavepackets in the 'gradient' torsion and metric background fields. To this end we go over to the Hamiltonian formalism for the action (3.2). The canonical momenta are $p_\mu = \partial L_{(\sigma)}/\partial v^\mu = -m\mathrm{e}^\sigma v_\mu/\sqrt{v^2}$, so the constraint is

$$\Pi_{(\sigma)} = p^2 - m^2\mathrm{e}^{2\sigma} = 0. \tag{3.7}$$

To construct the corresponding quantum field theory we can simply take the field action (1.5) with the new metric $g^{(\sigma)}_{\mu\nu}$ and canonically quantize it. The correspondence principle is automatically fulfilled. Indeed, in the semiclassical approximation for the quantum field theory associated with the action (1.5) the wavepackets would follow geodesics with respect to the background metric $g^{(\sigma)}_{\mu\nu}$ [23]. Making use of the symmetry (3.6) we see that the classical trajectories are autoparallels with respect to the metric $g_{\mu\nu}$ and the 'gradient' torsion generated by the background scalar field σ. Thus the scalar field action that leads to a quantum scalar field theory compatible with the correspondence principle is

$$S = \int \mathrm{d}^n q\, \mathrm{e}^{(n-2)\sigma}\sqrt{g}\big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\mathrm{e}^{2\sigma}\phi^2\big). \tag{3.8}$$

It yields the following equation of motion for the scalar field ϕ:

$$\Box\phi + (n-2)\,\partial_\mu\sigma\,\partial^\mu\phi + m^2\mathrm{e}^{2\sigma}\phi = 0 \tag{3.9}$$

where $\Box$ is the Laplace–Beltrami operator: $\Box\phi = (\sqrt{g})^{-1}\,\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi)$.

Equation (3.9) can be regarded as the quantum version of the constraint (3.7). Note that a multiplication of the constraint (3.7) by some function of coordinates would lead to an equivalent constraint on the classical level. In quantum theory the ordering of operators is generally not unique. Here we have promoted $\Pi_{(\sigma)}$ into an operator by multiplying it by $\mathrm{e}^{-2\sigma}$ and postulating that $\mathrm{e}^{-2\sigma}\hat{p}^2$ is the Laplace–Beltrami operator with respect to the metric $g^{(\sigma)}_{\mu\nu}$. This ensures the *hermiticity* of the constraint with respect to a scalar product with the measure $\sqrt{g^{(\sigma)}}\,\mathrm{d}^n q = \mathrm{e}^{n\sigma}\sqrt{g}\,\mathrm{d}^n q$, thus providing the *unitarity* of the time evolution.

4. Perturbation theory

Here we come to the conclusion that there is no Lagrangian formalism (subject to the physical conditions required) for the autoparallel motion, except for the gradient case discussed above. We make use of our third physical assumption that the Lagrangian, if it exists, should be analytic in the torsion tensor. So far, no experimental observation of torsion has been made. Therefore, the torsion force must be small compared with the gravitational force induced by the metric. This, in turn, suggests solving the integrability conditions for equation (2.7) by *perturbation* theory in the torsion tensor. We shall see that the integrability conditions are not fulfilled even in the first order of the perturbation theory, thus leading to the conclusion of the non-existence of the Lagrangian in general. We start with the ansatz

$$L(v, g, S) = L_g(v, g) + L_1(v, g, S) + \mathrm{O}(S^2) \tag{4.1}$$

which contains the Lagrangian L_g (1.3) for geodesics and a perturbation L_1 linear in the torsion tensor. From equation (2.7) it follows that the multiplier must also be analytic in torsion, so we set

$$\Omega_\mu{}^\lambda(v, g, S) = \delta^\lambda_\mu + \omega_\mu{}^\lambda(v, g, S) + \mathrm{O}(S^2). \tag{4.2}$$

In this approximation, the substitution of (2.6) in (2.7) leads to

$$[L_1]_\mu = \omega_\mu{}^\lambda[L_g]_\lambda + 2S_{\mu\nu\sigma}\frac{v^\nu v^\sigma}{\sqrt{v^2}}. \tag{4.3}$$

The variables $\dot{v}, v, q$ are considered as independent variables. The integrability conditions for (4.3) are still difficult to analyse because of the presence of the general functions ${\omega_\mu}^\lambda$. Therefore, we first look for the integrability conditions in the velocity space assuming the configuration space point to be fixed. We set $q^\mu = q_0^\mu$ after calculating all the derivatives ∂_μ in (4.3). Equation (4.3) is covariant under general coordinate transformations as a consequence of our second assumption. In particular, we may assume a geodesic coordinate system [24] at q_0^μ. This has the advantage that the Christoffel symbols are zero at the origin ${\overline{\Gamma}_{\mu\nu}}^\lambda(q_0) = 0$. Thanks to this property, the term ${\omega_\mu}^\lambda[L_g]_\lambda$ is proportional to the acceleration $\dot{v}^\mu$ and must cancel with the corresponding term contained in $[L_1]_\mu$. This leads to an equation for the multiplier which is not relevant for the subsequent analysis. For the remaining terms at $q^\mu = q_0^\mu$ we obtain

$$v^\nu \frac{\partial^2 L_1}{\partial q^\nu \partial v^\mu} - \frac{\partial L_1}{\partial q^\mu} = 2 S_{\mu\nu\sigma} \frac{v^\nu v^\sigma}{\sqrt{v^2}}. \tag{4.4}$$

Next, in the vicinity of q_0 we apply the Fourier transform $L_1(q, v) = \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kq} \tilde{L}_1(k, v)$, similarly for $\tilde{S}_{\mu\nu\sigma}(k)$, so that $\partial L_1/\partial q^\mu\big|_{q=q_0} = \int \mathrm{d}k\, \mathrm{i}k_\mu \mathrm{e}^{\mathrm{i}kq_0} \tilde{L}_1(k, v)$. Substituting this into (4.4), we obtain a first-order differential equation for $\tilde{L}_1(k, v)$ as a function of v^μ. This equation can be simplified by the ansatz $\tilde{L}_1 = k_\mu v^\mu c(k, v)$, leading to

$$\mathrm{i} \frac{\partial c}{\partial v^\mu} = \frac{2}{(k, v)^2} \tilde{S}_{\mu\nu\sigma} \frac{v^\nu v^\sigma}{\sqrt{v^2}}. \tag{4.5}$$

The integrability conditions for equation (4.5) are now easy to derive: $\partial^2 c/\partial v^\mu \partial v^\nu - \partial^2 c/\partial v^\nu \partial v^\mu = 0$. After multiplying them by the factor $\big((k, v)\sqrt{v^2}\big)^3$, they turn into a set of vanishing linear combinations of the monomials $v^\nu v^\sigma v^\alpha v^\beta$. Since v^μ are independent variables we are left with the equation

$$2k_{[\mu} \tilde{S}_{\lambda](\nu\sigma} \eta_{\alpha\beta)} + k_{(\nu} \{ \tilde{S}_{[\mu\lambda]\sigma} \eta_{\alpha\beta)} + \tilde{S}_{\sigma[\lambda\mu]} \eta_{\alpha\beta)} + \tilde{S}_{[\lambda|\alpha\beta} \eta_{\sigma)\mu]} \} = 0. \tag{4.6}$$

Here the indices $(\nu\sigma\alpha\beta)$ must be symmetrized, while the indices in the square brackets $[\mu\lambda]$ are antisymmetrized.

There are two cases where the integrability condition (4.6) is fulfilled identically and, hence, the Lagrangian always exists. First, we observe that $v^\mu \partial c/\partial v^\mu \equiv 0$ since $S_{\mu\nu\sigma} = -S_{\nu\mu\sigma}$. Therefore, c depends only on the angular variables in the velocity space, not on the norm $\sqrt{v^2}$. In two dimensions, equation (4.5) contains only one non-trivial equation which always has a solution. The Lagrangian can be constructed as proposed in appendix B. The second case is $\tilde{S}_{\mu\nu\lambda} \sim \delta^n(k)$, i.e. when the torsion tensor is constant in the coordinate system chosen. It is easy to obtain a simple recursion relation for an explicit form of all orders of perturbation theory for the Lagrangian L. However, the condition $\partial_\mu S_{\nu\lambda\sigma} = 0$ is not covariant under general coordinate transformations. So, the corresponding Lagrangian is not a scalar and cannot be regarded as physically acceptable.

The torsion tensor can always be decomposed into a trace, a totally antisymmetric part and a traceless part $Q_{\mu\nu\lambda}$ which is not totally antisymmetric. The totally antisymmetric part satisfies (4.6) identically because it does not contribute to the torsion force at all. We set

$${S_{\mu\nu}}^\sigma = \frac{1}{n-1}\big(S_\mu \delta_\nu^\sigma - S_\nu \delta_\mu^\sigma\big) + {Q_{\mu\nu}}^\sigma \tag{4.7}$$

where $S_\mu = {S_{\mu\lambda}}^\lambda$. Contracting (4.6) with $k^\nu k^\sigma k^\alpha k^\beta$, $\eta^{\nu\sigma}\eta^{\alpha\beta}$ and $k^\alpha k^\beta \eta^{\nu\sigma}$ we obtain a system of linear equations

$$\begin{aligned} &\beta_{\mu\lambda} + 3\gamma_{\mu\lambda} = 0 \qquad (2n+5)\alpha_{\mu\lambda} + (n+1)\beta_{\mu\lambda} = 0 \\ &3\alpha_{\mu\lambda} + (n+4)\beta_{\mu\lambda} + (2n+11)\gamma_{\mu\lambda} = 0 \end{aligned} \tag{4.8}$$

for $\alpha_{\mu\lambda} = k_{[\mu}\tilde{S}_{\lambda]}$, $\beta_{\mu\lambda} = k_\sigma(\tilde{S}_{[\mu\lambda]}{}^\sigma + \tilde{S}^\sigma{}_{[\lambda\mu]})$ and $\gamma_{\mu\lambda} = k_{[\mu}\tilde{S}_{\lambda]\nu\sigma}k^\nu k^\sigma/k^2$. As for every fixed μ and λ the determinant of the matrix of the coefficients is $2(n-2)(n+1)$, the system has only a trivial solution $\alpha_{\mu\nu} = \beta_{\mu\nu} = \gamma_{\mu\nu} = 0$ for $n > 2$. The relation $\alpha_{\mu\nu} = 0$ gives rise to a restriction on the trace S_μ

$$k_{[\mu}\tilde{S}_{\lambda]} = 0 \qquad \text{hence} \quad S_\lambda(q) \sim \partial_\lambda\sigma(q). \tag{4.9}$$

That is, in any dimension greater than two the contracted torsion tensor must be a potential vector field. At this point we may already conclude that for a *generic* torsion the inverse variational problem for the autoparallel equation has no solution. However, we proceed to analyse the traceless part of the torsion tensor. The 'gradient' part of the torsion tensor (4.7) satisfies (4.6) identically, so that the integrability condition (4.6) applies to $Q_{\mu\nu\sigma}$ only. We investigate it in three and four dimensions. Both cases are treated simultaneously.

The tensor $Q_{\mu\nu\sigma}$ can be parametrized in three and four dimensions, respectively, as

$$Q^{(3)}_{\mu\nu\lambda} = \epsilon_{\mu\nu\sigma}A^\sigma{}_\lambda \qquad Q^{(4)}_{\mu\nu\lambda} = \epsilon_{\mu\nu\sigma\rho}B^{\sigma\rho}{}_\lambda \tag{4.10}$$

where $A_{\mu\nu}$ is a symmetric, traceless 3×3 matrix (since $Q^{(3)\nu}_{\mu\nu} = 0$ and $\epsilon^{\mu\nu\sigma}Q^{(3)}_{\mu\nu\sigma} = 0$), and $B_{\sigma\rho\lambda}$ satisfies $\epsilon^{\mu\sigma\rho\lambda}B_{\sigma\rho\lambda} = 0$ and must be traceless $B^{\sigma\lambda}{}_\lambda = 0$ (since $Q^{(4)\nu}_{\mu\nu} = 0$ and $\epsilon^{\mu\nu\lambda\sigma}Q^{(4)}_{\nu\lambda\sigma} = 0$). Thus, $A_{\mu\nu}$ contains five independent components, while $B_{\mu\nu\sigma}$ has 16. They are subject to the conditions

$$k_\sigma\tilde{A}^{\sigma\rho} = 0 \qquad k_\sigma\tilde{B}^{[\mu\lambda]\sigma} = 0 \tag{4.11}$$

$$k_{(\nu}\tilde{A}_{\alpha\beta}\delta^\tau_{\sigma)} = 0 \qquad 2\eta_{(\alpha\beta|}k_\rho\tilde{B}^{\rho[\mu}{}_{|\sigma}\delta^{\lambda]}_{\nu)} + k_{(\nu}\tilde{B}_\beta{}^{[\mu}{}_\sigma\delta^{\lambda]}_{\alpha)} = 0. \tag{4.12}$$

Equation (4.11) is equivalent to $\beta_{\mu\nu} = 0$, while equation (4.12) stems from the integrability condition (4.6) where $\beta_{\mu\nu} = 0$ has been taken into account. It is possible to select five linearly independent equations for $A_{\mu\nu}$ and 16 for $B_{\mu\nu\sigma}$ out of these equations. Thus, we conclude that $Q^{(3,4)}_{\mu\nu\lambda} = 0$, and the Lagrangian exists only for the 'gradient' case.

5. Conclusions

We have proposed a rather general approach to study possible deviations from Einstein's equivalence principle due to the coupling between scalar matter and the torsion of spacetime. Our approach is based on the inverse problem of the calculus of variations and general principles of quantum field theory. It does not use any special generalization of Einstein's general relativity, neither does it rely on any specific form of the energy–momentum conservation law. It is far more general than the minimal gauge coupling principle which is typically used to construct a coupling between matter and spacetime geometry [2, 9]. We have shown that for a generic torsion, *no local* quantum field theory exists in four dimensions that leads to the autoparallel motion of spinless (scalar) particles in the semiclassical (eikonal) approximation. Only when the torsion tensor has a special form, the above problem admits a solution. In this case the coupling between scalar matter and torsion is described equivalently by the dilaton field whose existence is predicted by string theory [22].

The Einstein equivalence principle is not violated by the coupling between matter and the dilaton field if the coupling obeys the *universality* principle [25], meaning that it is constructed by the replacement $g_{\mu\nu} \to g^{(\sigma)}_{\mu\nu} = \mathrm{e}^{2\sigma}g_{\mu\nu}$ in the matter Lagrangian. Indeed, there would be no experiment that distinguishes the motion of test particles in the composite background metric $g^{(\sigma)}_{\mu\nu}$ from that in the background metric $g_{\mu\nu}$ and the dilaton field σ. A deviation from Einstein general relativity can only be seen in cosmology which is affected by *dynamics* of the dilaton field [25, 26].

We remark that the minimal gauge coupling principle does not predict the dilaton and leads only to the coupling between spin and torsion. As has been stressed by some authors (see the discussion in [27]) such a coupling might pose a consistency problem since the spin of composed particles is not simply a sum of the spins of its constituents, but also involves the orbital angular momentum. For instance, a spinning particle could be a bound state of *spinless* particles (e.g. a vector boson composed of a few scalar bosons, etc). Therefore, such a spinning particle would not interact with torsion at all. Thus, when applying the minimal gauge coupling principle, one has always to decide whether a given kind of particle is truly elementary or composite. Such a drawback could be circumvented either by allowing for the coupling of torsion to the angular momentum or by simply postulating that any theory for composite spinning particles should be consistent with the minimal gauge coupling principle, thus making a restriction on future fundamental theories. Given the difficulties of describing composite relativistic quantum fields, this latter option does not seem easy to pursue in practice, neither does it seem to admit a simple geometrical interpretation.

Here we have explored the first possibility. The autoparallel equation (1.1) ($F_\mu \equiv 0$) can be rewritten as the matter energy–momentum conservation law [28]

$$\overline{D}_\mu \overline{T}^{\mu\nu} + 2S^{\nu}{}_{\mu\sigma}\overline{T}^{\mu\sigma} = 0 \tag{5.1}$$

where $\overline{T}_{\mu\nu}$ is the energy–momentum tensor in general relativity (see also appendix A). The second term in (5.1) contains an interaction between the torsion and the angular momentum. We have proved that there exists no local quantum field theory whose dynamics complies with equation (5.1) except the special case when all the effects of torsion can be interpreted as those caused by the dilaton. One could also regard this result as an argument supporting the point of view that the spacetime geometry is specified only by the metric and possibly by the dilaton field.

We remark that the minimal price of incorporating equation (5.1) into a quantum theory is to give up locality [29]. This does not seem to us acceptable in quantum field theory of fundamental interactions, but still may be possible in effective theories describing a quantum motion of interstitial particles in crystals with topological defects.

Acknowledgments

We both acknowledge stimulating discussions with H Kleinert. SVS is grateful to F W Hehl for useful discussions and to M Henneaux for useful references. SVS was supported by DFG under the project Kl-25626-1.

Appendix A. Energy–momentum conservation for the autoparallels

The energy–momentum conservation law follows from the invariance of the action under general coordinate transformations. As compared with the geodesic action (1.3), the action (3.2) contains an extra scalar field describing the background spacetime geometry so that its variation is determined by both the variations of the metric $g_{\mu\nu}$ and the dilaton σ. Thus, we obtain

$$0 = \delta S = \int \mathrm{d}^4 q\, \sqrt{g}\left\{\frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu} + T^{\mu}{}_{\mu}\,\delta\sigma + \frac{\delta\mathcal{L}}{\delta q^\mu}\,\delta q^\mu\right\} \tag{A.1}$$

where, as usual,

$$T^{\mu\nu}(q) \equiv \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}(q)} = \frac{1}{\sqrt{g(q)}} \int \mathrm{d}s\, \delta^4(q - q(s)) p^\mu u^\nu \tag{A.2}$$

is the energy–momentum tensor and $\mathcal{L}$ is the Lagrangian density defined by $L = \int \mathrm{d}^3 q\, \mathcal{L}$. We observe that $T^{\mu\nu} = \mathrm{e}^\sigma \overline{T}^{\mu\nu}$ where $\overline{T}^{\mu\nu}$ is the energy–momentum tensor for the geodesic motion. The difference occurs through the σ dependence of the particle momentum: $p^\mu = -m\mathrm{e}^{\sigma(q)} u^\mu$. One can easily convince oneself that $\delta\mathcal{L}/\delta\sigma = T^\mu{}_\mu \equiv T$, which specifies the second term in (A.1). For the actual motion of the particle the third term in (A.1) vanishes and we obtain the energy–momentum conservation law (cf equation (5.1))

$$\overline{D}_\mu \overline{T}^{\mu\lambda} + \overline{T}^{\mu\lambda} \partial_\mu \sigma - \overline{T} \partial^\lambda \sigma = 0. \tag{A.3}$$

We see two additional terms occurring in the conservation law due to the torsion force. Integrating this equation over a three-dimensional spacelike hypersurface $q^0 =$ constant we again recover the autoparallel equation for the 'gradient' torsion (3.4).

Appendix B. The autoparallel Lagrangian in two dimensions

In two dimensions the integrability conditions (4.6) yield no restriction on torsion because of the time reparametrization invariance. Indeed, by fixing the gauge $q^0 \equiv t$ the problem becomes one dimensional. A general solution of the one-dimensional inverse variational problem was found by Darboux [30]. So, the Lagrangian always exists for the two-dimensional autoparallel motion. However, the constraint appears to be non-polynomial in the canonical momenta, thus leading to a *non-local* quantum field theory.

The torsion tensor can be parametrized in two dimensions by two scalar functions λ and σ:

$$S_{\mu\nu}{}^\alpha = \tfrac{1}{2} \epsilon_{\mu\nu} \left(\partial^\alpha \lambda + \epsilon^{\alpha\beta} \partial_\beta \sigma \right). \tag{B.1}$$

Let us decompose the velocity vector into the sum of two orthogonal vectors: $u^\mu = [\varphi k^\mu + (1 - \varphi^2)^{1/2} (\epsilon k)^\mu]/\sqrt{k^2}$, where $\varphi = (k, u)/\sqrt{k^2}$. Solving equation (4.5) for $c = c(\varphi)$ we find

$$\mathrm{i}\tilde{L}_1 = \sqrt{v^2} \left(\tilde{\sigma} + \varphi \ln[\varphi^{-1} + (\varphi^{-2} - 1)^{1/2}] \tilde{\lambda} \right). \tag{B.2}$$

The first term is the linear part of the Lagrangian (3.2) for the 'gradient' torsion. The second term is non-polynomial in φ. This property of the Lagrangian holds in higher orders of the perturbation expansion (4.1) as can be seen from a recursion relation for L_i. Therefore, the Lagrangian would lead to a constraint which is non-polynomial in p in all (but leading) orders of the perturbation theory. The corresponding quantum field theory will be non-local. We conclude that even in two dimensions only the 'gradient' torsion leads to an acceptable quantum field theory.

It is certainly possible to find a Lagrangian for the generic torsion (B.1). However, a complete discussion would be too involved and goes beyond the scope of this paper. Such a Lagrangian does not seem significant for physical applications. Just to give an idea of how the Lagrangian may look, we calculate it under the simplifying conditions $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_0 \lambda = 0$. This latter condition obviously violates the general coordinate invariance, but will allow us to find an *explicit* form of the Lagrangian by a short and simple method. We also set $\sigma = 0$ since the gradient case has already been discussed. After fixing the gauge by $q^0 = t$

($v^0 = 1$) and adopting the notations $v^1 \equiv v$, $\partial_1\lambda \equiv \partial_x\lambda$ we obtain one simple equation from the autoparallel equation (1.1)

$$\dot{v} + \partial_x\lambda(v - v^3) = 0. \tag{B.3}$$

The associated Lagrangian can be found via the Hamiltonian formalism. The first Hamiltonian equation is set to be $\dot{x} = p/\sqrt{1+p^2} = \omega\partial_p H$. Then the second Hamiltonian equation can be derived from (B.3) as $\dot{p} = -p\partial_x\lambda = -\omega\partial_x H$. These equations can easily be solved for the Hamiltonian H and the symplectic structure ω. Next, choosing Darboux coordinates $X = x$ and P such as $\partial_p P = \omega^{-1}$, the Lagrangian is obtained by the Legendre transformation for P: $L(v, \lambda) = P\dot{x} - H$. The time reparametrization invariance is restored by the rule $L_{(\lambda)}(v^0, v^1, \lambda) = v^0 L(v^1/v^0, \lambda)$ [14]. Therefore, the Lagrangian is

$$L_{(\lambda)} = -\sqrt{v^2}\cosh\lambda - v^1 \ln W \sinh\lambda \tag{B.4}$$

where $W = |v^0/v^1 + \sqrt{(v^0/v^1)^2 - 1}|$. It is not difficult to see that the constraint resulting from $u^2 = 1$ is not polynomial in the canonical momenta p_μ because $p_\mu = p_\mu(u)$ contain $\ln W$.

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